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О ВЛИЯНИИ ПОДГРУППЫ ФИТТИНГА НА ПРОИЗВЕДЕНИЯ КОНЕЧНЫХ РАЗРЕШИМЫХ ГРУПП

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ON THE INFLUENCE OF THE FITTING SUBGROUP ON THE PRODUCTS OF FINITE SOLUBLE GROUPS

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Подгруппа H группы G называется $\mathrm{F}(G)$-субнормальной, если она субнормальна в $H\mathrm{F}(G)$, где $\mathrm{F}(G)$ – подгруппа Фиттинга группы G. В работе описаны все замкнутые относительно взятия подгрупп Шмидта насыщенные формации $\mathfrak{F}$ конечных разрешимых групп, которые содержат всякую разрешимую группу G, представимую в произведение $\mathrm{F}(G)$-субнормальных $\mathfrak{F}$-подгрупп. Установлена сверхразрешимость группы G, содержащей три сверхразрешимые $\mathrm{F}(G)$-субнормальные подгруппы с попарно взаимно простыми индексами в G.

***Ключевые слова**: подгруппа Фиттинга, $\mathrm{F}(G)$-субнормальная подгруппа, сверхразрешимая группа, насыщенная формация, $\mathrm{F}(G)$-радикальная формация.*

A subgroup H of a group G is called $\mathrm{F}(G)$-subnormal if it is subnormal in $H\mathrm{F}(G)$. In the paper all closed under taking Schmidt subgroups saturated formations $\mathfrak{F}$ of finite soluble groups such that $\mathfrak{F}$ contains every soluble group G which is the product of its $\mathrm{F}(G)$-subnormal $\mathfrak{F}$-subgroups were described. It was shown that a group G which contains three supersoluble $\mathrm{F}(G)$-subnormal subgroups of pairwise coprime indexes in G is supersoluble.

***Keywords**: the Fitting subgroup, $\mathrm{F}(G)$-subnormal subgroup, supersoluble group, saturated formation, $\mathrm{F}(G)$-radical formation.*

Introduction

Only finite groups are considered. One of the important problems of group theory is the structural study of a finite group which can be factorized as a product of two or more pairwise permutable subgroups. The origin of this problem may be traced back to the well known theorem of Burnside about the solvability of biprimary groups.

Formations closed under taking products of certain types (arbitrary [1], normal and subnormal [2], abnormal and contrnormal [3] and etc.) of subgroups were studied in many papers. An important generalization of the subnormality is the $\mathfrak{F}$-subnormality [4], [5]. Formations closed under taking products of $\mathfrak{F}$-subnormal subgroups were studied in [6]–[8] etc. Formations with the Shemetkov property play significant role in this research. Recall that a formation $\mathfrak{F}$ is called a formation with the Shemetkov property if every s-critical group for $\mathfrak{F}$ is either a Schmidt group or a group of prime order.

In 1938 Fitting [9] showed that a product of two normal nilpotent subgroups is again nilpotent. It means that there exists the unique maximal normal nilpotent subgroup $\mathrm{F}(G)$ in every group G. This subgroup is called the Fitting subgroup. The Fitting subgroup has a great influence on the structure of a soluble group. That is why in the paper [10] authors introduced the following definition.

***Definition* 0.1.** *A subgroup H of a group G is called $\mathrm{F}(G)$-subnormal if H is subnormal in $H\mathrm{F}(G)$.*

A subnormal subgroup of a group G is obviously $\mathrm{F}(G)$-subnormal. The following example shows that in the general case a $\mathrm{F}(G)$-subnormal subgroup is not subnormal.

***Example* 0.2.** Let $G \simeq S_4$ be the symmetric group of degree 4. Let H be a Sylow 2-subgroup of G. Then H is a maximal subgroup of G which is not normal in G. Note that $\mathrm{F}(G) \subseteq H$. Hence H is $\mathrm{F}(G)$-subnormal in G. But H is not subnormal in G.

***Definition* 0.3.** *Let $\mathfrak{F}$ and $\mathfrak{X}$ be classes of soluble groups. We say that $\mathfrak{F}$ is $\mathrm{F}(G)$-radical in $\mathfrak{X}$ if $\mathfrak{F}$ is S_n-closed and contains every $\mathfrak{X}$-group $G = AB$ where A and B are $\mathrm{F}(G)$-subnormal $\mathfrak{F}$-subgroups of G.*

The following problem seems natural.

***Problem* A.** *Describe all classes (formations, Schunk classes, Fitting classes) of soluble groups which are $\mathrm{F}(G)$-radical in the class $\mathfrak{S}$ of all soluble groups.*

***Definition* 0.4.** *We shall call a class of groups $\mathfrak{X}$ S_{ch}-closed if $\mathfrak{X}$ contains with every group G all its Schmidt subgroups.*

Recall that $\mathfrak{S}_\pi$ is the class of all soluble π-groups. Formations closed under products of $\mathrm{F}(G)$-subnormal subgroups are described in the following theorem.

***Theorem* A.** *Let $\mathfrak{F}$ be a S_{ch}-closed saturated formation of soluble groups and $\pi = \pi(\mathfrak{F})$. The following statements are equivalent*:

(1) $\mathfrak{F}$ *is* F(G)-*radical in* $\mathfrak{S}$.

(2) $\mathfrak{F}$ *contains every soluble group* $G = AB$ *where* A *and* B *are* F(G)-*subnormal* $\mathfrak{F}$-*subgroups of* G.

(3) $\mathfrak{F}$ *is a hereditary formation and there exists a partition* $\sigma = \{\pi_i \mid i \in I\}$ *of* π *into mutually disjoint subsets such that* $\mathfrak{F} = \times_{i\in I} \mathfrak{S}_{\pi_i}$.

***Corollary* A.1** [10]. *Let* $G = AB$ *be a product of nilpotent* F(G)-*subnormal subgroups. Then G is nilpotent.*

***Corollary* A.2.** *Let* π *be a set of primes and a soluble group* $G = AB$ *be a product of* π-*decomposable* F(G)-*subnormal subgroups. Then G is* π-*decomposable.*

Let us note that the class

$$\times_{i\in I} \mathfrak{S}_{\pi_i} = (G \mid G = O_{\pi_{i_1}}(G)\times \ldots \times O_{\pi_{i_n}}(G))$$

is a lattice formation. Recall that a formation $\mathfrak{F}$ is called lattice if the intersection and the join of two $\mathfrak{F}$-subnormal subgroups is again a $\mathfrak{F}$-subnormal subgroup. This formations were studied by many researchers [5, chapter 6].

There are examples [11, p. 8] of non-supersoluble groups which are products of supersoluble normal (subnormal) subgroups. So the formation $\mathfrak{U}$ of all supersoluble groups is not F(G)-radical in $\mathfrak{S}$. R. Baer [12] showed that if a group G is the product of two normal supersoluble supgroups and G' is nilpotent then G is supersoluble. In [13] A.F. Vasil'ev and D.N. Simonenko generalized Baer's theorem on arbitrary hereditary saturated formations.

These results are the motivations for the following

***Problem* B.** *Let* $\mathfrak{X}$ *be a hereditary saturated formation of soluble groups. Describe all hereditary saturated* F(G)-*radical in* $\mathfrak{X}$ *subformations* $\mathfrak{F}$ *of* $\mathfrak{X}$.

***Theorem* B.** *Let* $\mathfrak{X}$ *be a hereditary saturated formation of soluble groups. The following statements are equivalent:*

(1) *Every hereditary saturated subformation* $\mathfrak{F}$ *of* $\mathfrak{X}$ *is* F(G)-*radical in* $\mathfrak{X}$.

(2) *Every group in* $\mathfrak{X}$ *has nilpotent derived subgroup.*

K. Doerk [14] showed that a group is supersoluble if it contains four supersoluble subgroups of pairwise coprime indexes. This result was generalized by O.U. Kramer [15] on arbitrary saturated formations of metanilpotent groups.

***Problem* C.** *Let* n *be a natural number*, $n \geq 3$ *and* $\mathfrak{F}$ *be a saturated formation of soluble groups such that* $\mathfrak{F}$ *contains every group G which has n* $\mathfrak{F}$-*subgroups of pairwise coprime indexes in G. Assume that a group G contains* $n-1$ F(G)-*subnormal* $\mathfrak{F}$-*subgroups of pairwise coprime indexes in G. Does* $G \in \mathfrak{F}$?

Partially answer on this problem is given in the following theorem.

***Theorem* C.** *Let* $\mathfrak{F}$ *be a hereditary saturated formation of metanilpotent groups with Sylow tower. If a group G contains three* F(G)-*subnormal* $\mathfrak{F}$-*subgroups of pairwise coprime indexes in G then* $G \in \mathfrak{F}$.

***Corollary* C.1.** *If a group G contains three* F(G)-*subnormal supersoluble subgroups of pairwise coprime indexes in G then G is supersoluble.*

***Corollary* C.2.** *Let* $\mathfrak{F}$ *be the formation of groups with nilpotent derived subgroup and Sylow tower. If a group G contains three* F(G)-*subnormal* $\mathfrak{F}$-*subgroups of pairwise coprime indexes in G then* $G \in \mathfrak{F}$.

D.K. Friesen [16] noted that if a group G contains two normal (subnormal) supersoluble subgroups of coprime indexes in G then G is supersoluble. The following example shows that we can not replace the subnormality by the F(G)-subnormality in Friesen's theorem.

***Example* 0.5.** Let a group G be isomorphic to the symmetric group of degree 3. Then there is a faithful irreducible G-module V of dimension 2 over F_7. Let T be the semidirect product of V and G. Consider $A = VG_3$ and $B = VG_2$ where G_p is a Sylow p-subgroup of G and $p \in \{2,3\}$. From $7 \equiv 1 (\mathrm{mod}\ p)$ and $p \in \{2,3\}$ it follows that A and B are supersoluble. Since V is a faithful irreducible G-module, F(T) = V. Now A and B are F(T)-subnormal supersoluble subgroups of T. Note that $T = AB$ is not supersoluble.

1 Preliminary results

We use standard notation and terminology that if necessary can be found in [17]. Recall some of them that are important in this paper. By $\mathbb{P}$ is denoted the set of all primes; $\pi(G)$ is the set of all prime divisors of the order of G; $\pi(\mathfrak{F}) = \bigcup_{G\in\mathfrak{F}} \pi(G)$; a group G is called π-group if $\pi(G) \subseteq \pi$; Z_p is the cyclic group of order p; $O_\pi(G)$ is the greatest normal π-subgroup of G; G' is the derived subgroup of G; $G^{\mathfrak{F}}$ is the $\mathfrak{F}$-residual for a formation $\mathfrak{F}$; $O_{p',p}(G)$ is the p-nilpotent radical of G for $p \in \mathbb{P}$ it also can be defined by $O_{p',p}(G)/O_{p'}(G) = O_p(G/O_{p'}(G))$; $\Phi(G)$ is the Frattini subgroup of G; $A \mathrm{wr} B$ is the regular wreath product of groups A and B; $G = N \rtimes M$ is the semidirect product of groups M and N ($N \triangleleft G$ and $N \cap M = 1$); $\mathfrak{G}_\pi$ ($\mathfrak{N}_\pi$) is the class of all (nilpotent) π-groups, where

$\pi \subseteq \mathbb{P}$. Let $\sigma = \{\pi_i \mid i \in I\}$ be a partition of π into mutually disjoint subsets then the class of all groups which are direct products of their (soluble) π_i-subgroups is denoted by $\underset{i\in I}{\times}\mathfrak{G}_{\pi_i}$ ($\underset{i\in I}{\times}\mathfrak{S}_{\pi_i}$).

Let $\mathfrak{F}$ and $\mathfrak{K}$ be formations then

$$\mathfrak{F}\mathfrak{K} = (G \mid G^{\mathfrak{K}} \in \mathfrak{F}).$$

A class of groups $\mathfrak{F}$ is called a formation if from $G \in \mathfrak{F}$ and $N \lhd G$ it follows that $G/N \in \mathfrak{F}$ and from $H/A \in \mathfrak{F}$ and $H/B \in \mathfrak{F}$ it follows that $H/A \cap B \in \mathfrak{F}$.

A class of groups $\mathfrak{X}$ is called hereditary (S_n-closed) if from $G \in \mathfrak{X}$ and $H \le G$ ($H \lhd G$) it follows that $H \in \mathfrak{X}$.

A class of groups $\mathfrak{X}$ is called weakly hereditary if from $p \in \pi(\mathfrak{X})$ it follows that $Z_p \in \mathfrak{X}$.

A class of groups $\mathfrak{X}$ is called saturated if from $G/\Phi(G) \in \mathfrak{X}$ it follows that $G \in \mathfrak{X}$.

A function $f : \mathbb{P} \to \{\mathit{formations}\}$ is called a formation function.

By well known Gashütz – Lubeseder – Schmid theorem saturated formations are exactly local formations, i. e. formations $\mathfrak{F} = LF(f)$ defined by a formation function f: $LF(f) = (G \in \mathfrak{G} \mid$ if H/K is a chief factor of G and $p \in \pi(H/K)$ then $G/C_G(H/K) \in f(p))$.

Among all local definitions of a local formation $\mathfrak{F}$ there is exactly one, denoted by F, such that F is integrated ($F(p) \subseteq \mathfrak{F}$ for all $p \in \mathbb{P}$) and full ($\mathfrak{N}_p F(p) = F(p)$ for all $p \in \mathbb{P}$). The function F is called the canonical local definition of $\mathfrak{F}$.

***Lemma* 1.1** [17, p. 357]. *Let f be a local definition of a formation $\mathfrak{F}$. A group G belongs $\mathfrak{F}$ if and only if $G/O_{p',p}(G) \in f(p)$ for all $p \in \pi(G)$.*

Recall some properties of Schmidt groups.

***Lemma* 1.2** [4, p. 243]. *Let G be a Schmidt group. Then*

(1) *$G = P \rtimes Q$ where P is the normal p-subgroup of G and Q is a cyclic Sylow q-subgroup of G that is not normal in G.*

(2) *$G/\Phi(G)$ is a Schmidt group.*

(3) *$P\Phi(G)/\Phi(G)$ is an elementary abelian p-subgroup and*

$$|Q\Phi(G)/\Phi(G)| = q .$$

Recall that $\mathfrak{F}^S$ is the greatest hereditary subclass of a class of groups $\mathfrak{F}$. Let $\mathfrak{X}$ be a class of groups. Recall that a group G is called s-critical for $\mathfrak{X}$ if $G \notin \mathfrak{X}$ but every proper subgroup of G belongs in $\mathfrak{X}$. The class of all s-critical for $\mathfrak{X}$ groups is denoted by $\mathcal{M}(\mathfrak{X})$. Note that $\mathcal{M}(\mathfrak{X}) = \mathcal{M}(\mathfrak{X}^S)$.

2 S_{ch}-closed formations

In the sequel a Schmidt (p, q)-group is a Schmidt $\{p, q\}$-group with a normal Sylow p-subgroup.

***Lemma* 2.1.** *Let $\mathfrak{F}$ be a saturated formation and S be a Schmidt (p, q)-group. If $S \in \mathfrak{F}$ then every Schmidt (p, q)-group belongs $\mathfrak{F}$.*

Proof. Let f be a local definition of $\mathfrak{F}$. From lemmas 1.1 and 1.2 it follows that $S/O_{p',p}(S) \simeq 1 \in f(p)$ and $S/O_{q',q}(S) \simeq Z_q \in f(q)$. Now if K is a Schmidt (p, q)-group then $K/O_{p',p}(K) \simeq 1$ and $K/O_{q',q}(K) \simeq Z_q$ by lemma 1.2. So $K \in \mathfrak{F}$ by lemma 1.1. □

***Theorem* 2.2.** *Let $\mathfrak{F} = LF(F)$ be a local formation of soluble groups and F be the canonical local definition of $\mathfrak{F}$. Then $\mathfrak{F}$ is S_{ch}-closed if and only if $F(p)$ is a weakly hereditary formation for every $p \in \pi(\mathfrak{F})$.*

Proof. Let $\mathfrak{F} = LF(F)$ be a S_{ch}-closed formation, F be the canonical local definition of $\mathfrak{F}$ and $p \in \pi(\mathfrak{F})$. Then $F(p) \neq \varnothing$.

Consider $q \in \pi(F(p))$. If $q = p$ then $Z_p \in \mathfrak{N}_p \subseteq F(p)$.

Assume that $q \neq p$. Let G be a group of minimal order such that $G \in F(p)$ and $q \in \pi(G)$. Note that $O_p(G) = 1$. Let $R = Z_p \mathrm{wr} G = L \rtimes G$ where $L = Z_p \times \ldots \times Z_p$ is the base of R.

From $G \in F(p)$, lemma 1.1 and the properties of the regular wreath product it follows that $R \in \mathfrak{F}$. Let R_q be a Sylow q-subgroup of R. Consider $T = LR_q$. By the properties of the regular wreath product $C_T(L) = L$. That is why T is nonnilpotent. Then T has a Schmidt (p, q)-subgroup S. So $S \in \mathfrak{F}$. Since $S/O_{p',p}(S) \simeq Z_q$, $Z_q \in F(p)$. Q.E.D.

Let $F(p)$ be a weakly hereditary formation for all $p \in \pi(\mathfrak{F})$. Assume that the theorem is false and let G be a minimal order counterexample. It means that $G \in \mathfrak{F}$ and G has a Schmidt (p, q)-subgroup $S \notin \mathfrak{F}$. Since G is soluble, we see that the order of every minimal normal subgroup of G is the power of a prime.

Let N be a minimal normal r-subgroup of G. Assume that $q \neq r$ and $p \neq r$. Then $N \cap S = 1$. It means that $S \simeq SN/N \subseteq G/N$. By our assumption $S \in \mathfrak{F}$, a contradiction.

Assume that $q = r$. From lemma 1.2 it follows that $N \cap S \le \Phi(S)$. It means that $SN/N \in \mathfrak{F}$. So SN/N is a Schmidt group by lemma 1.2. By lemma 2.1 $S \in \mathfrak{F}$, a contradiction.

Thus $O_{p'}(G)=1$. Now

$$q\in\pi(G/O_{p',p}(G))\subseteq\pi(F(p)).$$

From $S/O_{q',q}(S)=1$, $S/O_{p',p}(S)\simeq Z_q\in F(p)$ and lemma 1.1 it follows that $S\in\mathfrak{F}$, the final contradiction. □

***Lemma* 2.3.** *Every S_n-closed formation of soluble groups is weakly hereditary.*

Proof. Let $\mathfrak{F}$ be a S_n-closed formation of soluble groups, $G\in\mathfrak{F}$ and $p\in\pi(G)$. Since G is soluble, there is a chief factor H/K of G such that $\pi(H/K)=\{p\}$. Since $\mathfrak{F}$ is a S_n-closed formation, $H/K\in\mathfrak{F}$. From $H/K\simeq Z_p\times\ldots\times Z_p$ it follows that $Z_p\in\mathfrak{F}$. □

***Corollary* 2.4.** *Let $\mathfrak{F}$ be a saturated S_n-closed formation of soluble groups. Then $\mathfrak{F}$ is S_{ch}-closed.*

Proof. According to [17, p. 365] $\mathfrak{F}$ has the canonical local definition F such that $F(p)$ is a S_n-closed formation for every prime p. By lemma 2.3 $F(p)$ is a weakly hereditary formation for every prime p. By theorem 2.2 formation $\mathfrak{F}$ is S_{ch}-closed. □

The converse to corollary 2.4 is false. Let $\mathfrak{F}$ be the formation generated by the symmetric group S_4 of degree 4 and cyclic groups of orders 2 and 3. According to [18, p. 44] the alternating group A_4 of degree 4 does not belong $\mathfrak{F}$. It is well known that $\mathfrak{N}\mathfrak{F}$ is a local formation with the canonical local definition F where $F(p)=\mathfrak{N}_p\mathfrak{F}$ for all $p\in\mathbb{P}$. Since $\mathfrak{F}$ is a weakly hereditary formation, it is clear that $\mathfrak{N}_p\mathfrak{F}$ is also weakly hereditary for all $p\in\mathbb{P}$. By theorem 2.2 formation $\mathfrak{N}\mathfrak{F}$ is S_{ch}-closed. By theorem 10.3B [17] there is a faithful irreducible S_4-module V over F_7. Let $G=V\rtimes S_4$. Then $C_G(V)=V$ and $V=\mathrm{F}(G)$. It means that $G\in\mathfrak{N}\mathfrak{F}$. Note that $H=VA_4\lhd G$. Since $C_G(V)=V$, $O_{7',7}(H)=V$. From $H/O_{7',7}(H)\simeq A_4\notin\mathfrak{N}_7\mathfrak{F}$ it follows that $H\notin\mathfrak{N}\mathfrak{F}$. Thus $\mathfrak{N}\mathfrak{F}$ is a S_{ch}-closed but not S_n-closed formation.

***Theorem* 2.5.** *Let $\mathfrak{F}$ be a saturated S_{ch}-closed formation with the Shemetkov property. Then $\mathfrak{F}$ is a hereditary formation.*

Proof. Since $\mathfrak{F}$ is saturated, $\mathfrak{F}$ is weakly hereditary. Let us show that $\mathfrak{F}=\mathfrak{F}^S$. Assume that the set $\mathfrak{F}\setminus\mathfrak{F}^S\neq\varnothing$ and let G be a group of minimal order from it. Since $G\notin\mathfrak{F}^S$, there is a s-critical for $\mathfrak{F}^S$ subgroup H of G. Since $\mathcal{M}(\mathfrak{F})=\mathcal{M}(\mathfrak{F}^S)$, H is a s-critical for $\mathfrak{F}$ Schmidt group. From $G\in\mathfrak{F}$ it follows that $H\in\mathfrak{F}$, the contradiction. □

3 Final remarks and problems

Note that the $\mathrm{F}(G)$-subnormality is not a hereditary property, i. e. if H is a $\mathrm{F}(G)$-subnormal subgroup of a group G and $H\leq K\leq G$ then H is not $\mathrm{F}(K)$-subnormal in general. Also note that from the $\mathrm{F}(G)$-subnormality of H does not follow the $\mathrm{F}(G/N)$-subnormality of HN/N in G/N.

The main idea of the proof of theorem A (from (3) follows (1)) is to show that $\mathrm{F}(G)\leq Z_{\mathfrak{F}}(G)$. It was achieved by the result of [19] where the author showed that $Z_{\mathfrak{F}}(G)$ coincides with the intersection of all normalizers of all π_i-maximal subgroups of G for all $i\in I$ for any group G where $\mathfrak{F}=\times_{i\in I}\mathfrak{G}_{\pi_i}$. This result generalizes the well known theorem of R. Baer [20] that claims that the hypercenter of a group is the intersection of all normalizers of Sylow subgroups.

***Problem* 3.1.** *Describe all soluble $\mathrm{F}(G)$-radical in $\mathfrak{S}$ formations. Is there soluble non-saturated $\mathrm{F}(G)$-radical in $\mathfrak{S}$ formation?*

***Problem* 3.2.** *Describe all soluble (local) $\mathrm{F}(G)$-radical in $\mathfrak{S}$ Fitting classes.*

***Problem* 3.3.** *Is every soluble $\mathrm{F}(G)$-radical in $\mathfrak{S}$ Fitting class a formation?*

Recall [17, p. 453] that a Schunck class $\mathfrak{X}$ is called a *D*-class in a universe $\mathfrak{J}$ if every group G in $\mathfrak{J}$ has a unique class of maximal $\mathfrak{X}$-subgroups. We note that every Schunck *D*-class $\mathfrak{X}$ of soluble groups contains every soluble group $G=AB$ where A and B are $\mathrm{F}(G)$-subnormal $\mathfrak{X}$-subgroups of G.

In connection with this observation, the following problem seems to be interesting.

***Problem* 3.4.** *Describe all soluble $\mathrm{F}(G)$-radical in $\mathfrak{S}$ Schunck classes.*

Theorem *B* shows that the class of all groups with nilpotent derived subgroup is the greatest formation of soluble groups such that every its hereditary saturated subformation is $\mathrm{F}(G)$-radical in it.

***Problem* 3.5.** *Describe all saturated $\mathrm{F}(G)$-radical in $\mathfrak{N}^n$ formations.*

The two main ideas of the proof of theorem *C* is the induction on a Sylow tower and the following lemma:

Lemma [19]. *Let $\mathfrak{F}$ be the formation of all p-decomposable groups. Then $G^{\mathfrak{F}}=\langle[a,b]\mid a,b\in G,$ where a is a p-element, b is a q-element and $q\neq p\rangle$.*

***Problem* 3.6.** *Let a group G contain three $\mathrm{F}(G)$-subnormal metanilpotent subgroups with pairwise coprime indexes in G. Is G metanilpotent?*

In the universe of all groups there are a lot of groups G with $\mathrm{F}(G)=1$. In this universe the quasinilpotent radical $\mathrm{F}^*(G)$ and the Shemetkov – Schmid subgroup $\tilde{\mathrm{F}}(G)$ are the generalizations of the Fitting subgroup [21].

***Definition* 3.7.** *A subgroup H of a group G is called $\mathrm{F}^*(G)$-subnormal ($\tilde{\mathrm{F}}(G)$-subnormal) if H is subnormal in $H\mathrm{F}^*(G)$ ($H\tilde{\mathrm{F}}(G)$).*

***Definition* 3.8.** *Let $\mathfrak{F}$ be a class of groups. We say that $\mathfrak{F}$ is $\mathrm{F}^*(G)$-radical ($\tilde{\mathrm{F}}(G)$-radical) if $\mathfrak{F}$ is S_n-closed and contains every group $G = AB$ where A and B are $\mathrm{F}^*(G)$-subnormal ($\tilde{F}(G)$-subnormal) $\mathfrak{F}$-subgroups of G.*

It is natural to consider the following problems.

***Problem* 3.9.** *Describe all hereditary $\mathrm{F}^*(G)$-radical ($\tilde{F}(G)$-radical) formations.*

***Problem* 3.10.** *Is every hereditary $\mathrm{F}^*(G)$-radical ($\tilde{F}(G)$-radical) formation composition (saturated)?*

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